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Universal Perturbation Technique for Reanalysis of Non-Self-Adjoint Systems

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A novel matrix perturbation technique for modal dynamic reanalysis of non-self-adjoint systems is presented. This technique is established first by performing complex eigensubspace condensation and orthogonal decomposition procedure. The lower-order perturbations of eigensolutions (i.e., complex eigenvalues, complex left and right eigenvectors) are obtained by solving two reduced eigenvalue problems. The higher-order perturbations of eigensolutions are then derived by implementing the successive approximation process. The proposed technique is a kind of universal dynamic reanalysis method because it is universally applicable to systems with all of the three cases: distinct, repeated, and closely spaced eigenvalues. Illustrative examples are given to verify the present method, and satisfactory results are observed.

Nomenclature

c_{il}, d_{il}	= complex coefficients to be determined
Ex_i, Ey_i, $E\lambda_i$	= relative errors
I	= identity matrix of $(k - j + 1)$ order
$\bar{K}$	= $(k - j + 1) \times (k - j + 1)$ general matrix
K, K_0, K_1	= $n \times n$ general matrices
$k - j + 1$	= number of the repeated and/or closely spaced eigenvalues; see Eq. (3)
$\bar{M}$	= $(k - j + 1) \times (k - j + 1)$ general matrix
M, M_0, M_1	= $n \times n$ general matrices
n	= twice as large as the degrees of freedom of a system
p_i, q_i	= $(k - j + 1)$-dimensional column vectors to be determined
R, S	= $n \times (k - j + 1)$ complex eigensubspaces
x	= n-dimensional state vector
x_i	= ith right eigenvector
y_i	= ith left eigenvector
Δx_i, Δy_i, $\Delta\lambda_i$	= differences between the eigensolutions
δ_{ij}	= Kronecker delta
δx_i, δy_i	= n-dimensional column vectors of first order to be determined
ε	= small parameter
λ_i	= ith eigenvalue
μ_i	= approximation to λ_i with first-order precision
$\lvert\cdot\rvert$	= modulus of a complex number
$\perp$	= symbol indicating orthogonality

Subscripts

0, 1, 2, 3	= corresponding to the original solutions, the first-order, second-order, and third-order perturbations, respectively

Superscripts

H	= conjugate transpose
m_i	= index indicating the element of x_{i0} with the largest modulus
T	= transpose

Introduction

THE dynamic behavior of many physical systems is completely determined by the eigenvalues and eigenvectors of the systems. The changes in the system parameters give rise to changes in these eigensolutions and hence in dynamic response. As a result, in the dynamic design solving a generalized eigenvalue problem for any but the smallest systems is a major contributor to the computational expense. This is especially so when the so-called iterative design/analysis process is carried out. Furthermore, because solving a general non-self-adjoint system requires a state-space description where the complex left and right eigenvectors are needed for the decoupling, the dynamic reanalysis of non-self-adjoint systems is more computationally expensive and time consuming. Therefore, the perturbation analysis of an eigenvalue problem, which enables a quick evaluation of the changes of the dynamic characteristics as a result of the changes in the system parameters, is of prime importance and immense interest in many fields of physical science and engineering.

For non-self-adjoint systems with distinct complex eigenvalues, Courant and Hilbert first studied the perturbation of eigensolutions of a general matrix.[1] Meirovitch and Ryland developed a perturbation technique for gyroscopic systems with small internal damping and with external damping.[2,3] The derivatives of eigensolutions were investigated by Rogers,[4] Plaut and Huseyin,[5] Rudisill,[6] and Murthy and Haftka.[7] For non-self-adjoint systems with repeated eigenvalues, Liu and Chen proposed a matrix perturbation method.[8] Recently Liu improved this method.[9] From the perturbed eigensolutions, however, one can see that the method proposed in Ref. 9 does not possess the ability to deal with the case of closely spaced eigenvalues for non-self-adjoint systems. But the closely spaced eigenvalues quite often occur in many practical engineering systems, such as arbitrarily damped space structures, aeroelastic systems, control systems, large gyroscopic systems, and nearly periodic systems. On the other hand, for large complicated non-self-adjoint systems there frequently coexist three cases of eigenvalues, i.e., distinct, repeated, and closely spaced eigenvalues. Therefore, it is very necessary to develop a perturbation technique to be suitable for all of the three cases.

To meet the increasing demand for dynamic reanalysis of non-self-adjoint systems, a universal matrix perturbation technique is presented in this paper. This technique is rigorously developed and can be applied to all of the three cases of eigenvalues. Three numerical examples are given to demonstrate the validity of the universal perturbation technique.

The defective non-self-adjoint systems are not considered in the present paper.

Eigenvalue Problems and Normalization Conditions

The free vibration equations of motion for a general non-self-adjoint system with an n_0 degree-of-freedom system can be written in matrix form[3,9]:

$$\tilde{M}\ddot{q} + (\tilde{C} + \tilde{G})\dot{q} + (\tilde{K} + \tilde{H})q = 0 \tag{1a}$$

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where $\tilde{M}, \tilde{C}, \tilde{K}$ are the $n_0 \times n_0$ real symmetric mass, damping, stiffness matrices, respectively; $\tilde{G}, \tilde{H}$ are the $n_0 \times n_0$ real skew-symmetric gyroscopic, circulatory matrices, respectively; and q is the $n_0 \times 1$ column vector of generalized coordinates.

To explore the system characteristics, it is necessary to express Eq. (1a) in state-space form:

$$K_0 x = M_0 \dot{x} \tag{1b}$$

where $x = [\dot{q}^T, q^T]^T$ is the $n(=2n_0)$ dimensional state vector and

$$K_0 = \begin{bmatrix} -(\tilde{C} + \tilde{G}) & -(\tilde{K} + \tilde{H}) \\ I & 0 \end{bmatrix}, \qquad M_0 = \begin{bmatrix} \tilde{M} & 0 \\ 0 & I \end{bmatrix} \tag{1c}$$

or

$$K_0 = \begin{bmatrix} \tilde{M} & 0 \\ 0 & -(\tilde{K} + \tilde{H}) \end{bmatrix}, \qquad M_0 = \begin{bmatrix} 0 & \tilde{M} \\ \tilde{M} & \tilde{C} + \tilde{G} \end{bmatrix} \tag{1d}$$

The right and left eigenvalue problems of the original or unperturbed system represented by Eq. (1b) have the form

$$K_0 x_{i0} = \lambda_{i0} M_0 x_{i0}, \qquad i = 1, 2, \ldots, n \tag{2a}$$

$$K_0^T y_{i0} = \lambda_{i0} M_0^T y_{i0}, \qquad i = 1, 2, \ldots, n \tag{2b}$$

Without loss of generality, assume that the solution of Eqs. (2a) or (2b) produces repeated and/or closely spaced eigenvalues. These eigenvalues can be written as

$$\lambda_{j0} \cong \lambda_{j+1,0} \cong \cdots \cong \lambda_{k0} \tag{3}$$

which means that the original system possesses $(k - j + 1)$ repeated and/or closely spaced complex eigenvalues. The notation in Eq. (3) is nothing but for simplicity and convenience. If and only if both the real and imaginary parts of two eigenvalues are (nearly) equal, the two eigenvalues are (nearly) equal.

Upon normalization, the right and left eigenvectors satisfy the biorthogonality relation

$$y_{i0}^T M_0 x_{j0} = \delta_{ij}, \qquad i, j = 1, 2, \ldots, n \tag{4}$$

Obviously, Eq. (4) does not render the eigenvectors x_{i0} and y_{i0} unique. If the eigenvectors are not unique, neither are their perturbations. To obtain unique eigenvectors, another normalization condition has to be imposed. Herein we consider a simple and consistently effective normalizing condition

$$x_{i0}^{(m_i)} = 1, \qquad i = 1, 2, \ldots, n \tag{5}$$

where the index m_i can be chosen such that $|x_{i0}^{(m_i)}| = \max_{j=1\sim n} |x_{i0}^{(j)}|$ and $x_{i0}^{(m_i)}$, $x_{i0}^{(j)}$ are the m_ith and jth elements of x_{i0}, respectively.

The design changes in a system may be reflected by the changes in K_0 and M_0. Regardless of the reasons, the net effect is that K_0 and M_0 are changed. Because these changes are usually small compared to the entire system, the two updated matrices relative to that in Eq. (1b) can be expressed as

$$K = K_0 + \varepsilon K_1 \tag{6a}$$

$$M = M_0 + \varepsilon M_1 \tag{6b}$$

respectively, where εK_1 and εM_1 are the corresponding changes and are small relative to K_0 and M_0.

By recalling Eqs. (2), (4), and (5), and considering Eqs. (6), the perturbed eigenvalue problems have the form

$$(K_0 + \varepsilon K_1) x_i = \lambda_i (M_0 + \varepsilon M_1) x_i, \qquad i = 1, 2, \ldots, n \tag{7a}$$

$$(K_0 + \varepsilon K_1)^T y_i = \lambda_i (M_0 + \varepsilon M_1)^T y_i, \qquad i = 1, 2, \ldots, n \tag{7b}$$

and the corresponding normalization condition and biorthogonality property are

$$x_i^{(m_i)} = 1 \qquad i = 1, 2, \ldots, n \tag{8a}$$

$$y_i^T (M_0 + \varepsilon M_1) x_j = \delta_{ij} \qquad i, j = 1, 2, \ldots, n \tag{8b}$$

respectively.

Lower-Order Perturbations of Eigensolutions

For practical use the eigenvectors corresponding to the repeated and/or closely spaced eigenvalues are usually chosen to span two complex eigensubspaces:

$$S = [x_{j0}, x_{j+1,0}, \ldots, x_{k0}] \tag{9a}$$

$$R = [y_{j0}, y_{j+1,0}, \ldots, y_{k0}] \tag{9b}$$

In view of Eq. (4), we have

$$R^T M_0 S = I \tag{10}$$

Generally speaking, the variation between the original and perturbed eigenvectors may be significant, but the angle between the original and perturbed eigensubspaces can be regarded as small.[10] As a result, orthogonal decomposition of the perturbed eigenvectors x_i and y_i with respect to the original eigensubspaces can be made, i.e.,

$$x_i = Sp_i + \delta x_i \tag{11a}$$

$$y_i = Rq_i + \delta y_i \tag{11b}$$

$$\delta x_i \perp Sp_i \tag{11c}$$

$$\delta y_i \perp Rq_i \tag{11d}$$

in which $i = j, j+1, \ldots, k$. For the sake of space, the notation $i = j, j+1, \ldots, k$ will not be indicated thereinafter unless otherwise stated.

Substituting Eqs. (11a) and (11b) into Eqs. (7a) and (7b), using the variational principle, and considering Eqs. (11c) and (11d), we obtain two reduced eigenvalue problems of $(k - j + 1)$ order:

$$\bar{K} p_i = \mu_i \bar{M} p_i \tag{12a}$$

$$\bar{K}^T q_i = \mu_i \bar{M}^T q_i \tag{12b}$$

where

$$\bar{K} = R^T (K_0 + \varepsilon K_1) S \tag{13a}$$

$$\bar{M} = R^T (M_0 + \varepsilon M_1) S \tag{13b}$$

Inserting Eqs. (11a) and (11b) into Eqs. (8a) and (8b), using Eqs. (4) and (5), and neglecting second-order quantities, we have

$$(Sp_i)^{(m_i)} = 1 \tag{14a}$$

$$q_i^T \bar{M} p_i = 1 \tag{14b}$$

which are the normalization conditions corresponding to Eqs. (12a) and (12b), respectively.

Solving Eqs. (12) and (14) can give $(k - j + 1)$ groups of eigensolutions: μ_i, p_i, q_i. In most practical structures the number of the repeated eigenvalues and/or closely spaced eigenvalues is usually very small, i.e., $k - j + 1 \ll n$. In this case the solution procedure of Eqs. (12) and (14) will be more efficient. With the obtained $(k - j + 1)$ groups of eigensolutions, the lower-order perturbations of eigensolutions λ_i, x_i, and y_i can be determined [see Eqs. (11a) and (11b)]. Because there is usually only a small angle between the original and the corresponding perturbed eigensubspaces, Sp_i and Rq_i have errors of first order, and then μ_i has only an error of second order by the Rayleigh's quotient theorem.[10,11]

Higher-Order Perturbations of Eigensolutions

To find the higher-order perturbations, considering the accuracies of μ_i, Sp_i, and Rq_i just mentioned, the perturbed eigensolutions can be expressed as

$$\lambda_i = \mu_i + \varepsilon^2 \lambda_{i2} + \varepsilon^3 \lambda_{i3} \cdots \tag{15a}$$

$$x_i = Sp_i + \varepsilon x_{i1} + \varepsilon^2 x_{i2} + \cdots \tag{15b}$$

$$y_i = Rq_i + \varepsilon y_{i1} + \varepsilon^2 y_{i2} + \cdots \tag{15c}$$

Comparing Eqs. (15b) and (15c) with Eqs. (11a) and (11b), considering Eqs. (11c) and (11d), we have

$$\delta x_i = \varepsilon x_{i1} + \varepsilon^2 x_{i2} + \cdots \tag{16a}$$

$$\delta y_i = \varepsilon y_{i1} + \varepsilon^2 y_{i2} + \cdots \tag{16b}$$

$$x_{ir} \perp Sp_i, \qquad r = 1, 2, \ldots \tag{16c}$$

$$y_{ir} \perp Rq_i, \qquad r = 1, 2, \ldots \tag{16d}$$

Upon substituting Eqs. (15a) and (15b) into Eq. (7a), collecting coefficients of the same power of ε and neglecting fourth-order quantities, the subsequent equation can be decomposed as

$$[(K_0 + \varepsilon K_1) - \mu_i(M_0 + \varepsilon M_1)]Sp_i + \varepsilon(K_0 - \mu_i M_0)x_{i1} = 0 \tag{17a}$$

$$(K_1 - \mu_i M_1)x_{i1} - \lambda_{i2} M_0 Sp_i + (K_0 - \mu_i M_0)x_{i2} = 0 \tag{17b}$$

$$(K_1 - \mu_i M_1)x_{i2} - \lambda_{i3} M_0 Sp_i + (K_0 - \mu_i M_0)x_{i3} - \lambda_{i2}(M_0 x_{i1} + M_1 Sp_i) = 0 \tag{17c}$$

Considering Eq. (16c), we can expand x_{i1} and x_{i2} in the original right eigenvectors as follows:

$$x_{i1} = \sum_{l=1,\neq j\sim k}^{n} c_{il1} x_{l0} \tag{18a}$$

$$x_{i2} = \sum_{l=1,\neq j\sim k}^{n} c_{il2} x_{l0} \tag{18b}$$

Substituting Eq. (18a) into Eq. (17a), premultiplying the subsequent equation by y_{r0}^T, and using Eqs. (4) and (16c), we obtain

$$c_{il1} = \frac{y_{l0}^T (K_1 - \mu_i M_1) Sp_i}{\mu_i - \lambda_{l0}} \qquad l \neq j \sim k \tag{19}$$

and then

$$x_{i1} = \sum_{l=1,\neq j\sim k}^{n} \frac{y_{l0}^T (K_1 - \mu_i M_1) Sp_i}{\mu_i - \lambda_{l0}} x_{l0} \tag{20}$$

To compute λ_{i2}, inserting Eq. (18b) into Eq. (17b), premultiplying the subsequent equation by $(Rq_i)^T$, we have

$$q_i^T R^T (K_0 - \mu_i M_0) \sum_{l=1,\neq j\sim k}^{n} c_{il2} x_{l0} + q_i^T R^T (K_1 - \mu_i M_1) x_{i1} - \lambda_{i2} q_i^T R^T M_0 Sp_i = 0 \tag{21}$$

By using Eqs. (4) and (10), Eq. (21) reveals that

$$\lambda_{i2} = \frac{q_i^T R^T (K_1 - \mu_i M_1)}{q_i^T p_i} x_{i1} \tag{22}$$

Substituting Eq. (18b) into Eq. (17b), premultiplying the subsequent equation by y_{r0}^T, and using Eqs. (4) and (16c), we obtain c_{il2}, and then

$$x_{i2} = \sum_{l=1,\neq j\sim k}^{n} \frac{y_{l0}^T (K_1 - \mu_i M_1) x_{i1}}{\mu_i - \lambda_{l0}} x_{l0} \tag{23}$$

From Eqs. (18b), (17c), (4), and (10), we can show that

$$\lambda_{i3} = \frac{q_i^T R^T [(K_1 - \mu_i M_1)x_{i2} - \lambda_{i2} M_1 Sp_i]}{q_i^T p_i} \tag{24}$$

Working with Eqs. (15a), (15c), and (7b) results in

$$\left[(K_0 + \varepsilon K_1)^T - \mu_i (M_0 + \varepsilon M_1)^T\right] Rq_i + \varepsilon (K_0 - \mu_i M_0)^T y_{i1} = 0 \tag{25a}$$

$$(K_1 - \mu_i M_1)^T y_{i1} - \lambda_{i2} M_0^T Rq_i + (K_0 - \mu_i M_0)^T y_{i2} = 0 \tag{25b}$$

$$(K_1 - \mu_i M_1)^T y_{i2} - \lambda_{i3} M_0^T Rq_i + (K_0 - \mu_i M_0)^T y_{i3} - \lambda_{i2}\left(M_0^T y_{i1} + M_1^T Rq_i\right) = 0 \tag{25c}$$

To compute y_{i1}, y_{i2}, we first expand them in the original left eigenvectors:

$$y_{i1} = \sum_{l=1,\neq j\sim k}^{n} d_{il1} y_{l0} \tag{26a}$$

$$y_{i2} = \sum_{l=1,\neq j\sim k}^{n} d_{il2} y_{l0} \tag{26b}$$

Using the same pattern as just shown, we can obtain d_{il1}, d_{il2}, and then the perturbations of the left eigenvectors as follows:

$$y_{i1} = \sum_{l=1,\neq j\sim k}^{n} \frac{(Rq_i)^T (K_1 - \mu_i M_1) x_{l0}}{\mu_i - \lambda_{l0}} y_{l0} \tag{27a}$$

$$y_{i2} = \sum_{l=1,\neq j\sim k}^{n} \frac{y_{i1}^T (K_1 - \mu_i M_1) x_{l0}}{\mu_i - \lambda_{l0}} y_{l0} \tag{27b}$$

Up to this stage, we have obtained all of the unknown quantities in Eqs. (15a–15c). The even higher-order perturbations omitted herein for the sake of space and concision, even if with little use, can also be determined by the same procedure.

For the case of distinct eigenvalues, i.e., when $i \neq j, j+1, \ldots, k$, the proposed technique is also completely applicable. In fact, we only need to choose a single eigenvector to span the eigensubspaces in Eqs. (9), i.e.,

$$S = x_{l0}, \qquad l \neq j, j+1, \ldots, k \tag{28a}$$

$$R = y_{l0}, \qquad l \neq j, j+1, \ldots, k \tag{28b}$$

Similar but much simpler derivation can be made in the same way as just shown. In this case, for example, Eq. (12a) simplifies to the well-known Rayleigh's quotient, i.e., $\mu_l = \bar{K}/\bar{M}$.

As a result, the present technique is applicable to all of the three cases of distinct, repeated, and closely spaced eigenvalues. This is just the reason why we call it a universal technique.

Illustrative Examples

Example 1

Consider a single-degree-of-freedom mass (m)-spring (k)-damper (c) system represented by

$$m\ddot{u} + c\dot{u} + ku = 0$$

Introducing state vector $x = [\dot{u}, u]^T$ and letting $m = 1$, $k = 2$, and $c = 0.5$, we obtain the equations of motion of the original system in matrix notation:

$$K_0 x = M_0 \dot{x}$$

where

$$K_0 = \begin{pmatrix} -0.5 & -2 \\ 1 & 0 \end{pmatrix}, \qquad M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The two original eigenvalues are $-0.25 \pm 1.39194i$, where $i = \sqrt{-1}$. This belongs to the case of distinct eigenvalues. Suppose $m = 1$, $k = 2.1$, and $c = 0.6$ to produce the perturbed system, we have

$$\varepsilon K_1 = \begin{pmatrix} -0.1 & -0.1 \\ 0 & 0 \end{pmatrix}, \qquad \varepsilon M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

By using the present technique, the perturbed eigensolutions can be easily computed. To show the accuracy of the present technique, let us define the absolute errors of the perturbed eigensolutions as follows:

$$\Delta\lambda_i = (\lambda_i)_{\mathrm{PM}} - (\lambda_i)_{\mathrm{ES}} \tag{29a}$$

$$\Delta x_i = (x_i)_{\mathrm{PM}} - (x_i)_{\mathrm{ES}} \tag{29b}$$

$$\Delta y_i = (y_i)_{\mathrm{PM}} - (y_i)_{\mathrm{ES}} \tag{29c}$$

where the subscripts PM and ES refer to present method and exact solution, respectively. The relative errors are defined as

$$E\lambda_i = \frac{|\Delta\lambda_i|}{|(\lambda_i)_{\mathrm{ES}}|} \tag{30a}$$

$$Ex_i = \frac{\sqrt{(\Delta x_i)^H \Delta x_i}}{\sqrt{(x_i)^H_{\mathrm{ES}}(x_i)_{\mathrm{ES}}}} \tag{30b}$$

$$Ey_i = \frac{\sqrt{(\Delta y_i)^H \Delta y_i}}{\sqrt{(y_i)^H_{\mathrm{ES}}(y_i)_{\mathrm{ES}}}} \tag{30c}$$

For the sake of space, only the relative errors of the first-order and second-order perturbed eigensolutions are listed in Table 1.

Example 2

An eight-degree-of-freedom mass-spring-damper system is shown in Fig. 1. Introducing $x = [\dot{u}_1, \dot{u}_2, \ldots, \dot{u}_8, u_1, u_2, \ldots, u_8]^T$ and supposing $m_i = 1$, $k_i = 1$, $c_i = 0.5$, $(i = 1 \sim 8)$, $k_0 = 0$, and $c_0 = 0$, we obtain $K_0 x = M_0 \dot{x}$, where K_0 and M_0 are 16×16 non-symmetric matrices and are omitted herein for the sake of space. The 16 original eigenvalues are

$$-0.0301537 \pm 0.345985i, \qquad -0.0301537 \pm 0.345985i$$

$$-0.250000 \pm 0.968246i, \qquad -0.250000 \pm 0.968246i$$

$$-0.586824 \pm 1.41525i, \qquad -0.586824 \pm 1.41525i$$

$$-0.883022 \pm 1.65902i, \qquad -0.883022 \pm 1.65902i$$

in which there are eight groups of repeated eigenvalues.

Letting $m_i = 1$, $k_i = 1$, $c_i = 0.5$, $(i = 1 \sim 8)$, $k_0 = 0.1$, and $c_0 = 0.05$ yields the perturbed system. By using the present technique, the relative errors of the perturbed eigensolutions are listed in Table 2.

Example 3

The system shown in Fig. 1 is considered once more. Suppose that the parameters in the original system are $m_i = 1$, $c_i = 0.5$, $(i = 1 \sim 8)$, $k_i = 1$, $(i = 1 \sim 6, 8)$, $k_7 = 0.95$, $k_0 = 0$, and $c_0 = 0$. The 16 original eigenvalues are

$$-0.0301724 \pm 0.342963i, \qquad -0.0301537 \pm 0.345985i$$

$$-0.250000 \pm 0.968246i, \qquad -0.250000 \pm 0.968246i$$

$$-0.587316 \pm 1.40087i, \qquad -0.586824 \pm 1.41525i$$

$$-0.882512 \pm 1.64153i, \qquad -0.883022 \pm 1.65902i$$

Table 1 Relative errors of the two sets of perturbed eigensolutions (example 1)

Order	i	1	2
First	$E\lambda_i$	7.87E−4	7.87E−4
	Ex_i	4.46E−3	4.44E−3
	Ey_i	6.21E−3	6.20E−3
Second	$E\lambda_i$	1.54E−4	1.54E−4
	Ex_i	1.98E−4	2.10E−4
	Ey_i	2.20E−4	2.24E−4

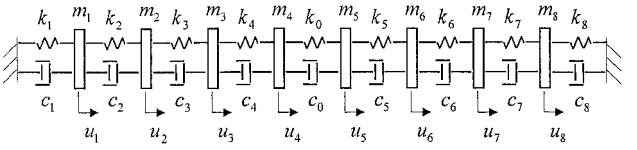

Fig. 1 Eight-degree-of-freedom mass-spring-damper system.

Table 2 Relative errors of the 16 sets of perturbed eigensolutions (example 2)

Order	i	1 (2)	3 (4)	5 (6)	7 (8)	9 (10)	11 (12)	13 (14)	15 (16)
First	$E\lambda_i$	0	5.77E−2	0	1.51E−3	0	6.67E−4	0	1.50E−4
		(0)	(5.77E−2)	(0)	(1.51E-3)	(0)	(6.67E−4)	(0)	(1.50E−4)
	Ex_i	2.85E−6	6.17E−2	1.29E−6	4.76E−3	1.84E−6	3.79E−3	1.52E−6	2.05E−3
		(1.63E−6)	(6.17E−2)	(3.33E−6)	(5.70E−3)	(1.67E−6)	(3.79E−3)	(1.48E−6)	(2.05E−3)
	Ey_i	3.78E−6	8.75E−2	2.76E−6	4.83E−3	3.52E−6	3.73E−3	3.78E−5	1.82E−3
		(1.53E−6)	(8.75E−2)	(2.78E−6)	(5.61E−3)	(4.53E−6)	(3.73E−3)	(3.76E−5)	(1.82E−3)
Second	$E\lambda_i$	0	1.44E−2	0	5.36E−5	0	4.51E−5	0	1.14E−5
		(0)	(1.44E−2)	(0)	(5.42E−5)	(0)	(4.51E−5)	(0)	(1.47E−5)
	Ex_i	2.84E−6	2.74E−2	1.29E−6	3.25E−3	1.84E−6	2.72−3	1.52E−6	1.37E−3
		(1.63E−6)	(2.81E−2)	(3.33E−6)	(3.46E−3)	(1.67E−6)	(2.71E−3)	(1.48E−6)	(1.36E−3)
	Ey_i	3.78E−6	4.19E−2	2.76E−6	3.39E−3	3.50E−6	2.68E−3	1.27E−6	1.53E−3
		(1.53E−6)	(4.32E−2)	(2.78E−6)	(3.49E−3)	(4.53E−6)	(2.76E−3)	(1.27E−6)	(1.58E−3)

Table 3 Relative errors of the 16 sets of perturbed eigensolutions (example 3)

Order	i	1 (2)	3 (4)	5 (6)	7 (8)	9 (10)	11 (12)	13 (14)	15 (16)
First	$E\lambda_i$	1.24E−5	5.82E−2	0	1.49E−3	1.58E−4	5.92E−4	7.52E−5	8.51E−5
		(1.24E−5)	(5.82E−2)	(0)	(1.49E−3)	(1.52E−4)	(5.92E−4)	(7.54E−5)	(8.51E−5)
	Ex_i	4.76E−3	5.96E−2	2.96E−6	4.69E−3	6.61E−2	6.34E−2	5.66E−3	1.90E−2
		(4.76E−3)	(5.96E−2)	(3.53E−6)	(4.08E−3)	(6.61E−2)	(6.34E−2)	(5.67E−3)	(1.88E−2)
	Ey_i	4.77E−3	8.57E−2	2.82E−6	4.76E−3	7.59E−2	7.33E−2	8.52E−2	1.37E−2
		(4.77E−3)	(8.57E−2)	(4.43E−6)	(4.11E−3)	(7.59E−2)	(7.34E−2)	(8.52E−2)	(1.36E−2)
Second	$E\lambda_i$	0	1.45E−2	0	5.31E−5	8.83E−6	5.01E−5	6.25E−6	6.40E−6
		(0)	(1.45E−2)	(0)	(5.31E−5)	(8.83E−6)	(5.01E−5)	(6.26E−6)	(6.91E−6)
	Ex_i	4.60E−4	2.90E−2	2.96E−6	3.46E−3	3.68E−3	5.32−3	2.32E−3	8.90E−3
		(4.61E−4)	(2.92E−2)	(3.53E−6)	(3.50E−3)	(3.65E−3)	(5.32E−3)	(2.33E−3)	(9.08E−3)
	Ey_i	4.52E−4	4.10E−2	2.76E−6	3.53E−3	4.12E−3	4.64E−3	1.65E−3	3.17E−3
		(4.52E−4)	(4.14E−2)	(4.43E−6)	(3.55E−3)	(4.13E−3)	(4.64E−3)	(1.65E−3)	(3.21E−3)

in which there are two groups of repeated eigenvalues and six groups of closely spaced eigenvalues.

The perturbed system is obtained by assuming $k_0 = 0.1, c_0 = 0.05$ with m_i, k_i, c_i, $(i = 1 \sim 8)$ unchanged. The relative errors of the perturbed eigensolutions are listed in Table 3.

Conclusion

The present method is presented to deal with the dynamic reanalysis for non-self-adjoint systems, especially for large complicated structures. This perturbation technique can greatly reduce the computational expense because it does not require a complete computation of the large-scale eigenvalue problem. From the results as shown in Tables 1–3, the following conclusions can be made.

In the three cases of eigenvalues, the first-order perturbed eigensolutions obtained by the present method have sufficient precision compared with the exact numerical solutions, and most of the second-order approximations, especially the second-order perturbed eigenvalues, are nearly equal to the exact solutions. Therefore, the present technique is an effective universal method. And, it can be argued that the present method will very likely give calculated results sufficiently accurate for engineering purposes in the case of large complicated structures.

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